Photometric Stereo: Lambertian Reflectance and Light Sources with Unknown Direction and Strength

by

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Abstract

This paper reconsiders the familiar case of photometric stereo under the assumption of Lambertian surface reflectance and three distant point sources of illumination. Here, it is assumed that the directions to and the relative strengths of the three light sources are not known *a priori*. Rather, estimation of these parameters becomes part of the problem formulation.

Each light source is represented by a 3-D vector that points in the direction of the light source and has magnitude proportional to the strength of the light source. Thus, nine parameters are required to characterize the three light sources. It is shown that, regardless of object shape, triples of measured intensity values are constrained to lie on a quadratic surface having six degrees of freedom. Estimation of the six parameters of the quadratic surface allows the determination of the nine parameters of the light sources up to an unknown rotation.

This is sufficient to determine object shape, although attitude with respect to the world-based or the camera-based coordinate system can not be simultaneously recovered without additional information.

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1 Introduction

Woodham [1] included a formulation of photometric stereo that allowed the recovery of both surface shape and surface reflectance under the assumptions of orthographic projection, three distant point sources of illumination and Lambertian reflectance. To apply the formulation given in [1], it is necessary that the three light sources be in a known configuration and be of known strength.

Subsequently, both for shape-from-shading and for photometric stereo, several papers have considered how the directions to the light sources might be recovered automatically. Automatic recovery is feasible using calibration points of known surface orientation. It is less clear how to proceed when the shape of the objects in view also is unknown. Recently, Iwahori *et al.* [2] argued that the three zenith angles of the illumination directions can be recovered in photometric stereo provided that the corresponding azimuth angles are known and that the illumination sources are of known strength.

Here, it is assumed that the directions to and the relative strengths of the three light sources are not known *a priori*. Rather, as with Iwahori *et al.* [2], estimation of these parameters becomes part of the problem formulation.

Each light source is represented by a 3-D vector that points in the direction of the light source and has magnitude proportional to the strength of the light source. Thus, nine parameters are required to fully characterize three light sources. It is shown that, regardless of object shape, triples of measured intensity values are constrained to lie on a quadratic surface having six degrees of freedom. Estimation of the six parameters of the quadratic surface allows one to determine the nine parameters of the light sources up to an unknown rotation.

This is sufficient to determine object shape, although attitude with respect to the world-based or the camera-based coordinate system can not be simultaneously recovered without additional information.

2 Formulation and Derivation of Theoretical Results

The basic equation characterizing image irradiance obtained under the assumptions of Lambertian reflectance, single distant point source illumination, orthographic projection and transmittance through an intervening scatterless medium is,

$$E(x,y) \;=\; \frac{E_0}{\pi}\,\rho(x,y)\,cos(\theta_i) \tag{1}$$

where $E(x, y)$ is the measured irradiance at image point (x, y), $\rho(x, y)$ is the bidirectional reflectance factor (aka albedo) at the corresponding object point $(0 \leq \rho(x, y) \leq 1)$, E_0 is the irradiance of the light source and θ_i is the angle of incidence of the light source at the object point measured.

Often, one assumes that $\rho(x, y)$ is constant at all object points of interest so that the dependence of ρ on (x, y) can be ignored. (Note, however, that Woodham [1] does consider the case where ρ depends on (x, y).) Then, one can also assume, without loss of generality, that the constant of proportionality, $\frac{E_0}{\pi}\rho$, can be taken equal to 1 so that the image irradiance equation becomes the more familiar

$$E(x, y) \;=\; cos(\theta_i) \tag{2}$$

Here, however, we will want to allow the relative strengths of the light sources to be distinct so that we will write

$$E(x, y) \;=\; E\; cos(\theta_i) \tag{3}$$

Since we are not interested in absolute units, we will refer to the parameter E as the relative strength of the light source.

The dependence of image irradiance, $E(x, y)$, on the cosine of the incident angle, θ_i, makes the Lambertian case especially simple to analyze. If we represent directions by unit vectors, then the cosine of the angle between any two directions is the dot product of the corresponding two unit vectors. This allows a linear problem formulation, as will be exploited here.

For three light source photometric stereo, let $\mathbf{a}_i \;=\; [a_{i1}, a_{i2}, a_{i3}]$, $i \;=\; 1, 2, 3$, be the 1×3 (row) vectors that point in the direction of light source i with magnitude equal to the relative strength, E_i, of light source i. Let $\mathbf{A}$ be the 3×3 matrix

$$\mathbf{A} \;=\; \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{4}$$

We will assume that the three light source directions, given by $\mathbf{a}_i$, $i \;=\; 1, 2, 3$, are not coplanar so that the matrix $\mathbf{A}$ is nonsingular.

Let $\mathbf{x} \;=\; [x_1, x_2, x_3]^T$ be the unit (column) surface normal vector at some object point of interest. Let $\mathbf{y} \;=\; [y_1, y_2, y_3]^T$ be the associated triple of intensity values given by Equation (3), applied once for each light source direction. Then, we may write

$$\mathbf{y} \;=\; \mathbf{A}\,\mathbf{x} \tag{5}$$

Equation (5) establishes a linear relation between surface shape, given by the unit surface normal vector, $\mathbf{x}$, and measured intensity values, $\mathbf{y}$.

Of course, if we knew $\mathbf{A}$ then we could determine $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{B}\,\mathbf{y} \tag{6}$$

where $\mathbf{B} = \mathbf{A}^{-1}$. Here, however, we do not assume that $\mathbf{A}$ is known. Fortunately, there is more that we can say simply based on the observation that Equation (5) is linear.

Consider each unit vector, $\mathbf{x}$, to be positioned at the origin. We can then associate all vectors, $\mathbf{x}$, with points on the unit sphere centered at the origin. In this way, we can think of Equation (5) as specifying a linear transformation of the sphere $\| \mathbf{x} \|_2 = (\mathbf{x}^T\mathbf{x})^{\frac{1}{2}} = 1$. It is reasonable to ask what is the corresponding shape mapped out by the vectors $\mathbf{y} = \mathbf{A}\,\mathbf{x}$.

Substitution, using Equation (6), shows that the quadratic $\mathbf{x}^T\mathbf{x} = 1$ implies

$$(\mathbf{B}\,\mathbf{y})^T\,\mathbf{B}\,\mathbf{y} = \mathbf{y}^T\,\mathbf{B}^T\mathbf{B}\,\mathbf{y} = \mathbf{y}^T\,\mathbf{C}\,\mathbf{y} = 1 \tag{7}$$

where $\mathbf{C} = \mathbf{B}^T\mathbf{B}$ is the 3×3 symmetric, positive definite matrix

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1{}^T\mathbf{b}_1 & \mathbf{b}_1{}^T\mathbf{b}_2 & \mathbf{b}_1{}^T\mathbf{b}_3 \\ \mathbf{b}_2{}^T\mathbf{b}_1 & \mathbf{b}_2{}^T\mathbf{b}_2 & \mathbf{b}_2{}^T\mathbf{b}_3 \\ \mathbf{b}_3{}^T\mathbf{b}_1 & \mathbf{b}_3{}^T\mathbf{b}_2 & \mathbf{b}_3{}^T\mathbf{b}_3 \end{bmatrix} \tag{8}$$

and where the $\mathbf{b}_i = [b_{1i}, b_{2i}, b_{3i}]^T$, $i = 1, 2, 3$, are the three 3×1 column vectors of $\mathbf{B}$.

Suppose we were to obtain an empirical scatterplot of measured intensity triples, $\mathbf{y}$, from points on an object of unknown shape. Then, these intensity triples are constrained to lie on the quadratic $\mathbf{y}^T\,\mathbf{C}\,\mathbf{y} = 1$. Geometrically, this means that the intensity triples, $\mathbf{y}$, lie on an ellipsoid whose equation is given analytically by

$$c_{11}\,{y_1}^2 + c_{22}\,{y_2}^2 + c_{33}\,{y_3}^2 + 2\,c_{12}\,y_1\,y_2 + 2\,c_{13}\,y_1\,y_3 + 2\,c_{23}\,y_2\,y_3 - 1 = 0 \tag{9}$$

This equation has only six unknown coefficients. This follows, of course, from the fact that the matrix $\mathbf{C}$, being symmetric, has only six degrees of freedom. Equation (9) necessarily defines an ellipsoid because the matrix $\mathbf{C}$ is positive definite. In particular, $c_{ii} > 0$, $i = 1, 2, 3$.

The six unknown coefficients of matrix $\mathbf{C}$ can be determined empirically even when the matrix $\mathbf{A}$ is unknown and even when there is no object point whose surface normal vector, $\mathbf{x}$, is known. All that is required is that we have sufficiently many measured intensity triples, $\mathbf{y}$, to estimate the six unknown coefficients of the quadratic given by Equation (9).

A standard linear least squares method can be used to estimate these six coefficients. Let $\mathbf{y}_k = [y_{k1}\ y_{k2}\ y_{k3}]$, $k = 1, 2, \ldots, n$, be $n \gg 6$ measured intensity triples. Let $\mathbf{M}$ be the $n \times 6$ matrix

$$\mathbf{M} = \begin{bmatrix} {y_{11}}^2 & {y_{12}}^2 & {y_{13}}^2 & 2\,y_{11}y_{12} & 2\,y_{11}y_{13} & 2\,y_{12}y_{13} \\ {y_{21}}^2 & {y_{22}}^2 & {y_{23}}^2 & 2\,y_{21}y_{22} & 2\,y_{21}y_{23} & 2\,y_{22}y_{23} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {y_{n1}}^2 & {y_{n2}}^2 & {y_{n3}}^2 & 2\,y_{n1}y_{n2} & 2\,y_{n1}y_{n3} & 2\,y_{n2}y_{n3} \end{bmatrix} \tag{10}$$

Let $\mathbf{I}$ be the $n \times 1$ column vector each entry of which is 1. Let $\mathbf{z}$ be the 6×1 column vector of unknown coefficients, $\mathbf{z} = [c_{11}\ c_{22}\ c_{33}\ c_{12}\ c_{13}\ c_{23}]^T$. Then, the standard linear least squares estimate of $\mathbf{z}$ is

$$\mathbf{z} = (\mathbf{M}^T\mathbf{M})^{-1}\,\mathbf{M}^T\,\mathbf{I} \tag{11}$$

In principle, this least squares estimation will be very robust since each of the n measured intensity triples contributes useful information.

Thus, empirical measurement determines the matrix $\mathbf{C}$. The constraint that $\mathbf{C}$ imposes on $\mathbf{A}$ is easiest to interpret when expressed in terms of $\mathbf{C}^{-1}$. Let $\mathbf{D} = \mathbf{C}^{-1}$ so that

$$\mathbf{D} = \mathbf{C}^{-1} = (\mathbf{B}^T\mathbf{B})^{-1} = \mathbf{B}^{-1}(\mathbf{B}^T)^{-1} = \mathbf{B}^{-1}(\mathbf{B}^{-1})^T = \mathbf{A}\mathbf{A}^T \tag{12}$$

Therefore,

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1{\mathbf{a}_1}^T & \mathbf{a}_1{\mathbf{a}_2}^T & \mathbf{a}_1{\mathbf{a}_3}^T \\ \mathbf{a}_2{\mathbf{a}_1}^T & \mathbf{a}_2{\mathbf{a}_2}^T & \mathbf{a}_2{\mathbf{a}_3}^T \\ \mathbf{a}_3{\mathbf{a}_1}^T & \mathbf{a}_3{\mathbf{a}_2}^T & \mathbf{a}_3{\mathbf{a}_3}^T \end{bmatrix} \tag{13}$$

The matrix $\mathbf{D}$, like the matrix $\mathbf{C}$, is a 3×3 symmetric, positive definite matrix. From $\mathbf{D}$, one can determine the relative strengths of the light sources i, $i = 1, 2, 3$, and the angle between the vectors to light sources i and j, $i \neq j$, $i = 1, 2, 3$, $j = 1, 2, 3$. Specifically, the relative strength of light source i, E_i, is given by

$$E_i = (\mathbf{a}_i{\mathbf{a}_i}^T)^{\frac{1}{2}} = \sqrt{d_{ii}} \tag{14}$$

and the cosine of the angle, α_{ij}, $i \neq j$, between $\mathbf{a}_i$ and $\mathbf{a}_j$ is given by

$$cos(\alpha_{ij}) = \frac{\mathbf{a}_i{\mathbf{a}_j}^T}{\sqrt{\mathbf{a}_i{\mathbf{a}_i}^T}\,\sqrt{\mathbf{a}_j{\mathbf{a}_j}^T}} = \frac{d_{ij}}{\sqrt{d_{ii}}\,\sqrt{d_{jj}}} \tag{15}$$

Equations (14) and (15) together represent six constraints on the matrix $\mathbf{A}$. These six constraints can be interpreted geometrically. Let the vectors $\mathbf{a}_i$, $i = 1, 2, 3$, share a common origin. The vectors, $\mathbf{a}_i$, form a triad whose shape, specified by the lengths of the vectors and the angles between them, is known. Any rotation of this triad will not change the shape of the triad and therefore will not violate any of the six constraints. A 3-D rotation has three degrees of freedom. To absolutely fix the triad in a given 3-D coordinate system, three additional constraints would be required.

A simple argument demonstrates that the quadratic $\mathbf{y}^T\,\mathbf{C}\,\mathbf{y} = 1$ is, indeed, invariant under a rotation of the coordinate system used to represent $\mathbf{x}$. Let $\mathbf{R}$ be an arbitrary 3×3 rotation matrix. Consider rotating our unit surface normals by $\mathbf{R}$. That is, let $\hat{\mathbf{x}} = \mathbf{R}\,\mathbf{x}$ Clearly, the constraint $\hat{\mathbf{x}}^T\hat{\mathbf{x}} = 1$ is preserved since

$$\hat{\mathbf{x}}^T\hat{\mathbf{x}} = (\mathbf{R}\,\mathbf{x})^T\mathbf{R}\,\mathbf{x} = \mathbf{x}^T(\mathbf{R}^T\mathbf{R})\mathbf{x} = \mathbf{x}^T\mathbf{x} = 1 \tag{16}$$

Therefore, the corresponding constraint $\mathbf{y}^T\,\mathbf{C}\,\mathbf{y} = 1$ also is preserved. Suppose we have

$$\hat{\mathbf{y}} = \mathbf{A}\hat{\mathbf{x}} \tag{17}$$

We can also write this as

$$\hat{\mathbf{y}} = \mathbf{A}\,(\mathbf{R}\,\mathbf{x}) = (\mathbf{A}\,\mathbf{R})\,\mathbf{x} \tag{18}$$

Thus, given only measurements $\mathbf{y}$, one can not distinguish between the case of a surface normal vector $\hat{\mathbf{x}}$ with light source matrix $\mathbf{A}$ from the case of a surface normal vector $\mathbf{x}$ with light source matrix $\hat{\mathbf{A}} = \mathbf{A}\,\mathbf{R}$. That is, without knowing $\mathbf{x}$, one can only hope to determine the matrix $\mathbf{A}$ up to an unknown rotation matrix $\mathbf{R}$.

Shape, in the form of surface normal vectors, still can be reconstructed using a coordinate system imposed for that purpose. The relationship between the surface normals thus reconstructed and any world-based or object-based coordinate system will be one of rotation. Actually, the argument holds for any orthonormal matrix $\mathbf{R}$, not just a rotation. It is not possible, based on matrix $\mathbf{C}$ alone, to decide whether the triad $\mathbf{a}_i$, $i = 1, 2, 3$, forms a right-handed or a left-handed coordinate system. If a mismatch occurs between the handedness of the coordinate system used for reconstruction and the handedness of the final coordinate system, the possibility that $\mathbf{R}$ includes a reflection must also be considered.

Assuming that matrix $\mathbf{C}$ has been determined empirically, any assignment of values to the entries of matrix $\mathbf{A}$ that satisfies Equation (13) is suitable since this assignment will implicitly define an 3-D coordinate system in which to reconstruct the surface normal vectors, $\mathbf{x}$. The following procedure selects a candidate matrix $\hat{\mathbf{A}}$ that satisfies $\mathbf{C}^{-1} = \mathbf{D} = \hat{\mathbf{A}}\,\hat{\mathbf{A}}^T$:

1. Align $\mathbf{a}_1$ with the positive X_1 axis by setting $a_{12} = a_{13} = 0$. To satisfy Equation (14), set $a_{11} = \sqrt{d_{11}}$.

2. Place $\mathbf{a}_2$ in the X_1X_2 plane by setting $a_{23} = 0$. To satisfy Equation (15) for $\mathbf{a}_1$ and $\mathbf{a}_2$, set $a_{21} = d_{21}/a_{11}$ To satisfy Equation (14), set $a_{22} = \pm\sqrt{d_{22} - a_{21}{}^2}$. The $\pm$ on a_{22} is significant because $\mathbf{a}_2$ may make an angle α_{12} with respect to the X_1 axis either in a clockwise or in a counter clockwise direction.

3. To satisfy Equation (15) for $\mathbf{a}_1$ and $\mathbf{a}_3$, set $a_{31} = d_{31}/a_{11}$. To satisfy Equation (15) for $\mathbf{a}_2$ and $\mathbf{a}_3$, set $a_{32} = (d_{32} - a_{21}a_{31})/a_{22}$ To satisfy Equation (14), set $a_{33} = \pm\sqrt{d_{33} - a_{31}{}^2 - a_{32}{}^2}$. Again, the $\pm$ on a_{33} is significant because $\mathbf{a}_3$ may be oriented either in the half space $X_3 > 0$ or the half space $X_3 < 0$

Depending on which sign is chosen for a_{22} and a_{33} above, the triad $\mathbf{a}_i$, $i = 1, 2, 3$, will form a left-handed or a right-handed system. A right-handed (left-handed) system can always be chosen according to whether the resulting determinant of $\hat{\mathbf{A}}$ is positive (negative).

Given the matrix, $\hat{\mathbf{A}}$, the surface normal, $\hat{\mathbf{x}}$, corresponding to a measured intensity triple, $\mathbf{y}$ is given by:

$$\hat{\mathbf{x}} = (\hat{\mathbf{A}})^{-1}\,\mathbf{y} \tag{19}$$

The surface normal, $\hat{\mathbf{x}}$, is expressed in the 3-D coordinate system imposed by our particular construction of $\hat{\mathbf{A}}$.

Now, if the visual task is object recognition, this may well be sufficient. There is no particular advantage, for viewpoint independent object recognition, to representing surface normals in a viewer-centered coordinate system. On the other hand, if the task requires a robot to grasp the object, it will be necessary to determine object attitude in a robot-centered coordinate system.

Suppose other means are used to determine the rotation matrix, $\mathbf{R}$, that transforms surface normals, $\mathbf{x}$, represented in a desired coordinate system to the coordinate system implicitly defined by the method used to construct $\hat{\mathbf{A}}$. Now,

$$\hat{\mathbf{x}} = \mathbf{R}\,\mathbf{x} = (\hat{\mathbf{A}})^{-1}\,\mathbf{y} \tag{20}$$

so that

$$\mathbf{y} = \hat{\mathbf{A}}\,\mathbf{R}\,\mathbf{x} \tag{21}$$

and

$$\mathbf{A} = \hat{\mathbf{A}}\,\mathbf{R} \tag{22}$$

Clearly, we can also recover $\mathbf{x}$ from $\hat{\mathbf{x}}$ by

$$\mathbf{x} = \mathbf{R}^{-1}\,\hat{\mathbf{x}} = \mathbf{R}^T\,\hat{\mathbf{x}} \tag{23}$$

3 An Example

Let the three unit (row) vectors pointing in the direction of the light sources be

$$\begin{array}{lccc} [& .5568900989 & .2386671853 & .7955572842 &] \\ [& -.5568900989 & .2386671853 & .7955572842 &] \\ [& 0 & 0 & 1 &] \end{array} \quad (24)$$

Suppose the relative strengths of the three light sources, E_i, $i = 1, 2, 3$, are 3, 2 and 1.5. Then, the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} 1.670670297 & .7160015559 & 2.386671853 \\ -1.113780198 & .4773343706 & 1.591114568 \\ 0 & 0 & 1.500000000 \end{bmatrix} \quad (25)$$

The matrix $\mathbf{C} = \mathbf{B}^T\mathbf{B}$, where $\mathbf{B} = \mathbf{A}^{-1}$, becomes

$$\mathbf{C} = \begin{bmatrix} .5772234818 & .5971277402 & -1.551827789 \\ .5971277402 & 1.298752835 & -2.327741684 \\ -1.551827789 & -2.327741684 & 5.382716048 \end{bmatrix} \quad (26)$$

Figure 1 shows three 256×256 synthesized images of a Lambertian sphere illuminated from each of the three light source directions given in Equation (24). For the images shown in Figure 1, the light sources were assumed to be of equal strength. In what follows, the intensity values were post-multiplied, respectively, by 3, 2 and 1.5 to correspond to the situation described by Equation (25). For this example, $24,812$ points on the sphere received illumination from all three light sources. Using Equation (11), with $n = 24,812$, the empirically determined estimate of $\mathbf{C}$, call it $\hat{\mathbf{C}}$, is given by

$$\hat{\mathbf{C}} = \begin{bmatrix} .5769297 & .5967482 & -1.5509174 \\ .5967482 & 1.2980919 & -2.3263760 \\ -1.5509174 & -2.3263760 & 5.3797137 \end{bmatrix} \quad (27)$$

The input images shown in Figure 1 were quantized to 8 bits. Comparison of the matrices $\mathbf{C}$ and $\hat{\mathbf{C}}$ indicates that each entry of $\hat{\mathbf{C}}$ is accurate to better than 3 decimal digits (i.e., better than 10 bits). This illustrates the inherent robustness of the estimate, $\hat{\mathbf{C}} \approx \mathbf{C}$, when based on a large number of measurement triples. Now, let $\mathbf{D} = \mathbf{C}^{-1}$ so that

$$\mathbf{D} = \begin{bmatrix} 9 & 2.278481010 & 3.580007761 \\ 2.278481010 & 4 & 2.386671844 \\ 3.580007761 & 2.386671844 & 2.25 \end{bmatrix} \quad (28)$$

From the square roots of the diagonal elements of $\mathbf{D}$ we immediately recover the relative strengths of the three light sources, E_i, $i = 1, 2, 3$, as 3, 2 and 1.5, respectively. Following the procedure outlined above, a candidate matrix, $\hat{\mathbf{A}}$, is obtained where

$$\hat{\mathbf{A}} = \begin{bmatrix} 3 & 0 & 0 \\ .7594936718 & 1.850180893 & 0 \\ 1.193335923 & .8001059613 & .4310218286 \end{bmatrix} \tag{29}$$

It can be verified that the matrix $\hat{\mathbf{A}}$ satisfies the constraints. That is, $\hat{\mathbf{A}}\,\hat{\mathbf{A}}^T = \mathbf{D} = \mathbf{C}^{-1}$. In this example, it happens that both $\mathbf{A}$ and $\hat{\mathbf{A}}$ define a right-handed coordinate system. In this example, we also know $\mathbf{A}$ so that we can determine $\mathbf{R} = (\hat{\mathbf{A}})^{-1}\mathbf{A}$ as

$$\mathbf{R} = \begin{bmatrix} .5568901001 & .2386671858 & .7955572859 \\ -.8305861935 & .1600211927 & .5334039755 \\ 0 & -.9578262846 & .287347886 \end{bmatrix} \tag{30}$$

Suppose we obtain a measured intensity triple, $\mathbf{y}$, where

$$\mathbf{y} = \begin{bmatrix} 2.755891272 & .5511782542 & .8660254035 \end{bmatrix} \tag{31}$$

Then, we solve for $\hat{\mathbf{x}} = (\hat{\mathbf{A}})^{-1}\mathbf{y}$ to obtain

$$\hat{\mathbf{x}} = \begin{bmatrix} .9186304258 & -.0791899548 & -.387100880 \end{bmatrix} \tag{32}$$

The corresponding $\mathbf{x} = \mathbf{R}^T\,\hat{\mathbf{x}}$ is

$$\mathbf{x} = \begin{bmatrix} .5773502706 & .5773502652 & .5773502721 \end{bmatrix} \tag{33}$$

This completes the example since the surface normal $\mathbf{x} = \left[1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right]$ was used to generate $\mathbf{y}$ in Equation (31) according to $\mathbf{y} = \mathbf{A}\,\mathbf{x}$ and $1/\sqrt{3} \approx 0.57735027$.

4 Conclusions

Nine parameters are required to characterize the light sources for the case of three distant point source photometric stereo. For the Lambertian case, it has been shown that, regardless of object shape, triples of measured intensity values determine six of these nine parameters.

The remaining three degrees of freedom define a rotation. For the purposes of shape reconstruction, it is not necessary to know this rotation.

A simple, but illustrative, application of this result would be to robot navigation. Imagine a robot vehicle required to move through a large, unstructured workspace. Suppose that the positions of three distant point light sources are known in a 3-D coordinate system defined in the workspace. Further, suppose the robot were to carry a suitable target object of known shape in its field of view. The method described here can be used to reconstruct the shape of the target object in a light source based coordinate system. Given that the position and attitude of the target already is known in the robot coordinate system, the rotation matrix, $\mathbf{R}$, between light source and robot coordinates also can be determined. This, in turn, defines the attitude of the robot with respect to the workspace coordinate system.

In this work, two unique properties of Lambertian reflectance have been exploited. First, a Lambertian reflector appears equally bright from all viewing directions. This means that it is not essential to reconstruct object shape in a viewer dependent coordinate system, as is the case with standard approaches to shape-from-shading and photometric stereo. Instead, as has been demonstrated, one can reconstruct shape in a 3-D coordinate system induced by the light sources themselves. Second, under distant point source illumination, Lambertian reflectors have scene radiance proportional to the cosine of the incident angle. This means that, using unit vectors to specify directions, the problem remains linear. The resulting mathematics, as has been demonstrated, remains quite tractable.

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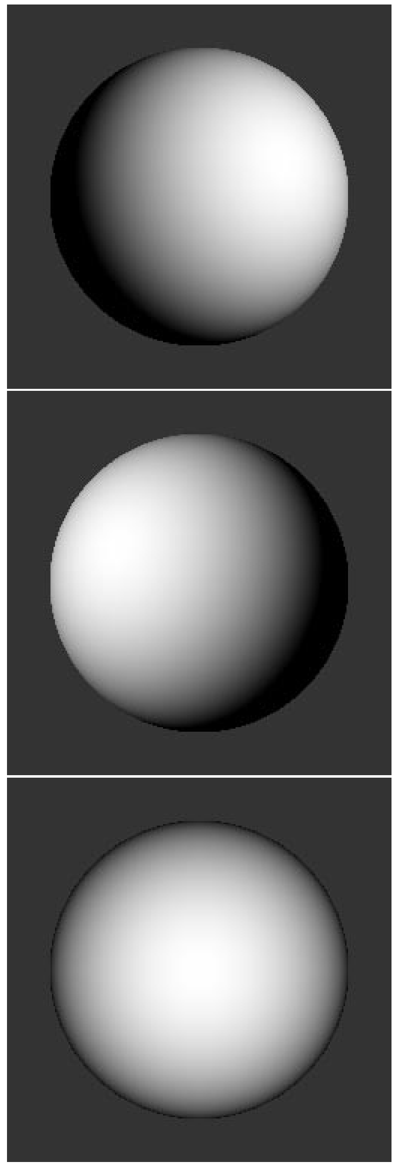

Figure 1